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C Neumann and Bargmann systems associated with the coupled KdV soliton hierarchy

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Abstract. Under two different constraints between the potentials and the eigenfunctions, the eigenvalue problem associated with the coupled KdV hierarchy is nonlinearised to be a completely integrable C Neumann system on the tangent bundle of sphere TS^{N-1} with the Hamiltonian

$$H^* = -\tfrac{1}{2}\langle\Lambda q, p\rangle - \tfrac{1}{2}\langle q, q\rangle\langle p, p\rangle + \tfrac{1}{2}\langle q, p\rangle^2 + \tfrac{1}{2}\langle q, p\rangle\langle\Lambda p, p\rangle$$

and a completely integrable Bargmann system $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q, H)$ with the Hamiltonian

$$H = -\tfrac{1}{2}\langle\Lambda q, p\rangle + \tfrac{1}{2}\langle p, p\rangle\langle q, p\rangle - \tfrac{1}{2}\langle q, q\rangle$$

respectively. The involutive solutions of the coupled KdV equation associated with the two systems are given.

1. Introduction

It is an important task to search for new finite-dimensional completely integrable systems. A general method has been developed in [1, 2, 7], through which integrable systems are obtained by the 'nonlinearisation' of eigenvalue problems associated with given soliton hierarchies.

Consider the eigenvalue problem

$$L(u)\psi_j = \lambda_j\psi_j \qquad \text{or} \qquad \partial_x\psi_j = M(u, \lambda_j)\psi_j \tag{1.1}$$

generally used in soliton theory. They are linear when the coefficient or potential $u(x)$ is given. In the case when $u(x)$ is an N-soliton potential (Bargmann's potential) or finite-band potential (the C Neumann's potential), it can often be expressed as a polynomial or other elementary function of eigenfunctions $\psi = (\psi_1, \ldots, \psi_N)$: $u = f(\psi)$; in addition, a constraint condition $g(\psi) = 0$ is satisfied in the Neumann case. Therefore, what the eigenfunctions actually satisfy is a system of nonlinear ordinary differential equations:

$$L(f(\psi))\psi = \Lambda\psi \qquad \text{or} \qquad \partial_x\psi = M(f(\psi), \Lambda)\psi \tag{1.2}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, with the constraint condition $g(\psi) = 0$ in the Neumann case.

It is interesting that (1.2) has been successfully verified to be a completely integrable system in the Liouville sense [6] for quite a few eigenvalue problems associated with soliton hierarchies [1, 2, 7]. The verification is not trivial in each case, where the inner structure of the eigenvalue problem and its isospectral equations is involved.

In the process of nonlinearisation of (1.1) into (1.2), a central role is played by the relation $u=f(\psi)$, and the constraint $g(\psi)=0$ in the Neumann case, which are difficult to obtain. A possible expression is suggested in [2], where $u=f(\psi)$ and $g(\psi)=0$ can be solved from

$$G_0=\sum_{j=1}^{N}\beta_j\frac{\delta\lambda_j}{\delta u} \qquad \text{and} \qquad G_{-1}=\sum_{j=1}^{N}\gamma_j\frac{\delta\lambda_j}{\delta u}$$

respectively, in which G_{-1} and G_0 are the first two Lenard gradients. Though our proof is not general, it does work for quite a few soliton hierarchies.

In this paper we are going to investigate the coupled KdV [3] (cKdV) hierarchy, which is the isospectral hierarchy of the eigenvalue problem:

$$y_x=\begin{pmatrix}-\frac{1}{2}\lambda+\frac{1}{2}u & -v\\ 1 & \frac{1}{2}\lambda-\frac{1}{2}u\end{pmatrix}y \qquad y=\begin{pmatrix}y_1\\ y_2\end{pmatrix}. \tag{1.3}$$

The nonlinearisation of (1.3) gives two kinds of finite-dimensional completely integrable systems, whose involutive solution is mapped into a solution of the cKdV equation by the nonlinearisation relation $(u,v)=f(p,q)$.

Let the time evolution of the eigenfunction y of (1.3) obey the differential equation:

$$y_t=A(G)y-\lambda B(G)y \tag{1.4}$$

with $G=(G^{(1)},G^{(2)})^{\mathrm{T}}$ and

$$A(G)=\begin{pmatrix}\frac{1}{2}(G_x^{(2)}+uG^{(2)}) & G_x^{(1)}-vG^{(2)}\\ G^{(2)} & -\frac{1}{2}(G_x^{(2)}+uG^{(2)})\end{pmatrix}$$

$$B(G)=\frac{1}{2}\begin{pmatrix}G^{(2)} & 0\\ 0 & -G^{(2)}\end{pmatrix}.$$

The compatible condition for (1.3) and (1.4) gives the evolution equation:

$$w_t=KG-\lambda JG \qquad w=(u,v)^{\mathrm{T}} \tag{1.5}$$

where the Lenard operator pair K, J are $(\partial=\partial/\partial x)$

$$K=\begin{pmatrix}2\partial & \partial^2+\partial u\\ -\partial^2+u\partial & v\partial+\partial v\end{pmatrix} \qquad J=\begin{pmatrix}0 & \partial\\ \partial & 0\end{pmatrix}.$$

Consider the Lenard gradients G_j defined recursively by

$$\begin{cases}KG_{j-1}=JG_j & j=-1,0,1,2,\ldots\\ G_{-2}=(1,0)^{\mathrm{T}} & G_{-1}=(0,1)^{\mathrm{T}}.\end{cases} \tag{1.6}$$

G_j is a polynomial of u, v and their derivatives, which is uniquely determined if the constant terms $aG_{-2}+bG_{-1}$ are agreed to be zero for $j\geqslant 0$. Let

$$G=\sum_{j=-1}^{m}G_{j-1}\lambda^{m-j} \tag{1.7}$$

then (1.5) is reduced to the soliton equation:

$$w_t=JG_m=KG_{m-1}.$$

$X_j=JG_j$ is the cKdV vector field, the first few being:

$$X_{-2}=X_{-1}=0 \qquad X_0=\begin{pmatrix}u_x\\ v_x\end{pmatrix} \qquad X_1=\begin{pmatrix}u_{xx}+2uu_x+2v_x\\ -v_{xx}+2(uv)_x\end{pmatrix}.$$

Let c_j be constants, then

$$w_t = X_m + c_1 X_{m-1} + \ldots + c_m X_0$$

is called the higher-order CKdV equation.

Let λ_j and $y(x) = (q_j(x), p_j(x))^T$ be the eigenvalue and the associated eigenfunction of (1.3). Denote the functional gradient

$$\nabla_{(u,v)}\lambda = (\delta\lambda/\delta u, \delta\lambda/\delta v)^T$$

with regard to the bilinear form defined by the integral

$$\langle \xi, \eta \rangle = \int_\Omega \xi\eta \, dx$$

by $\nabla\lambda$ for short, where Ω is $(-\infty, \infty)$ or $(-T, T)$ in the decaying at infinity condition or the periodic condition, respectively. We have:

Proposition 1.1.

$$K\nabla\lambda_j = \lambda_j J \nabla\lambda_j \qquad \nabla_{(u,v)}\lambda_j = (-p_j q_j, p_j^2)^T. \tag{1.8}$$

2. A completely integrable C Neumann system

Consider the Neumann constraint (see [2])

$$G_{-1} = \sum_{j=1}^{N} \nabla\lambda_j \qquad \text{i.e.} \qquad \langle q, p \rangle = 0 \quad \langle p, p \rangle = 1 \tag{2.1}$$

where $q = (q_1, \ldots, q_N)^T$, $p = (p_1, \ldots, p_N)^T$, $\langle \cdot, \cdot \rangle$ denote the standard product in $\mathbb{R}^N$. By differentiating (2.1) with respect to x and using (1.1), we obtain

$$u = \langle \Lambda p, p \rangle \qquad v = \langle q, q \rangle \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N). \tag{2.2}$$

The nonlinearisation of (1.3) under (2.2) yields the Neumann system ($' = \partial/\partial x$):

$$(\mathrm{C}) \qquad \begin{cases} q' = -\frac{1}{2}\Lambda q + \frac{1}{2}\langle \Lambda p, p \rangle q - \langle q, q \rangle p & (2.3) \\ p' = q + \frac{1}{2}\Lambda p - \frac{1}{2}\langle \Lambda p, p \rangle p & (2.4) \\ \langle p, p \rangle = 1 \end{cases}$$

where (2.4) and $\langle p, p \rangle = 1$ imply $\langle q, p \rangle = 0$, which can be verified by substituting (2.4) into $\langle p, p' \rangle = 0$.

Proposition 2.1. The functions defined by $(m = 0, 1, 2, \ldots)$

$$F_0 = -\tfrac{1}{2}\langle \Lambda q, p \rangle - \tfrac{1}{2}\langle q, q \rangle \langle p, p \rangle + \tfrac{1}{2}\langle q, p \rangle^2 \tag{2.5}$$

$$F_m = -\tfrac{1}{2}\langle \Lambda^{m+1} q, p \rangle - \tfrac{1}{2} \sum_{i+j=m} \begin{vmatrix} \langle \Lambda^i q, q \rangle & \langle \Lambda^i q, p \rangle \\ \langle \Lambda^j p, q \rangle & \langle \Lambda^j p, p \rangle \end{vmatrix} \tag{2.6}$$

are in involution in pairs, that is the Poisson bracket $(F_k, F_l) = 0$ in the symplectic space $(\mathbb{R}^{2N}, dp \wedge dq)$.

Proof. See [4, 5].

Theorem 2.2. The system $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q, F_m)$ is completely integrable in the Liouville sense.

Discuss the Moser constraint on the tangent bundle

$$\mathrm{TS}^{N-1} = \{(p, q) \in \mathbb{R}^{2N} \mid F = \langle q, p\rangle = 0, G = \tfrac{1}{2}(\langle p, p\rangle - 1) = 0\}.$$

Through direct calculations we have

$$(F, F_m) = 0 \qquad (F, G) = 1 \qquad (F_m, G) = -\tfrac{1}{2}\langle \Lambda^{m+1} p, p\rangle.$$

Thus the Lagrangian multipliers are

$$\mu_m = \frac{(F_m, G)}{(F, G)} = -\tfrac{1}{2}\langle \Lambda^{m+1} p, p\rangle. \tag{2.7}$$

Notice $F = 0$ on the tangent bundle TS^{N-1}, hence the restriction of the canonical equation of $H^* = F_0 - \mu_0 F$ on TS^{N-1} is

$$\begin{cases} q' = F_{0,p} - \mu_0 F_p|_{\mathrm{TS}^{N-1}} = -\tfrac{1}{2}\Lambda q + \tfrac{1}{2}\langle \Lambda p, p\rangle q - \langle q, q\rangle p \\ p' = -F_{0,q} + \mu_0 F_q|_{\mathrm{TS}^{N-1}} = \tfrac{1}{2}\Lambda p + q - \tfrac{1}{2}\langle \Lambda p, p\rangle p \\ \langle p, p\rangle = 1 \end{cases} \tag{2.8}$$

which is exactly the Neumann system (C).

Theorem 2.3. The Neumann system defined by (C) $(\mathrm{TS}^{N-1}, \mathrm{d}p \wedge \mathrm{d}q|_{\mathrm{TS}^{N-1}}, H^* = F_0 - \mu_0 F)$ is completely integrable in the Liouville sense.

Proof. Let $F_m^* = F_m - \mu_m F$, $m = 1, 2, \ldots, N-1$, then it is easy to verify that $(F_k^*, F_l^*) = 0$ on TS^{N-1}. Hence $\{F_m^*\}$ is an involutive system.

Theorem 2.4. Let (q, p) be a solution of the Neumann system (C), then $u = \langle \Lambda p, p\rangle$, $v = \langle q, q\rangle$ satisfy a stationary cKdV equation

$$X_N + \alpha_1 X_{N-1} + \ldots + \alpha_N X_0 = 0 \tag{2.9}$$

with suitably chosen constants $\alpha_1, \ldots, \alpha_N$.

Proof. Acting with the operator $(J^{-1}K)^k$ upon the first formula of (2.1), we get

$$G_k + \sum_{j=1}^{k+1} c_j G_{k-j} + \varepsilon_k G_{-2} = \sum_{j=1}^{N} \lambda_j^{k+1} \nabla \lambda_j \tag{2.10}$$

in view of (1.6), (1.8) and $\ker J = \{aG_{-1} + bG_{-2} | \forall a, b\}$. Consider the polynomial $(\beta_0 = 1)$:

$$P(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_N) = \beta_0 \lambda^N + \beta_1 \lambda^{N-1} + \ldots + \beta_N. \tag{2.11}$$

Acting with the operator $J \sum_{k=1}^{N} \beta_{N-k} \cdot$ on (2.10), we have (2.9).

3. The involutive solution of the cKdV equation associated with the Neumann system

Consider the canonical system of the F_m^* flow on the tangent bundle TS^{N-1}:

$$(F_m^*) \quad \frac{\partial}{\partial t_m}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial F_m^*/\partial p \\ -\partial F_m^*/\partial q \end{pmatrix} = I \nabla F_m^* \qquad I = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{3.1}$$

where I_N is the $N\times N$ unit matrix. Denote the solution operator of the initial-value problem by $g_m^{t_m}$, then its solution can be expressed as

$$\begin{pmatrix} q(t_m) \\ p(t_m) \end{pmatrix} = g_m^{t_m}\begin{pmatrix} q(0) \\ p(0) \end{pmatrix}.$$

Since any two F_k^*, F_l^* are involutive, we have [6]:

Proposition 3.1. (*a*) Any two canonical systems (F_k^*) and (F_l^*) are compatible; (*b*) the Hamiltonian phase flows $g_k^{t_k}$, $g_l^{t_l}$ commute.

Denote the flow variables of (F_0^*) and (F_m^*) by $x=t_0$, $t=t_m$, respectively, and define

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} = g_0^x g_m^{t_m}\begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}$$

The commutativity of g_0^x, $g_m^{t_m}$ implies that there exists a smooth function of (x, t_m), which is called the involutive solution of the consistent system of equations (F_0^*), (F_m^*).

Theorem 3.2. Let $(q(x,t_m), p(x,t_m))^{\mathrm{T}}$ be an involutive solution of the consistent system (F_0^*), (F_m^*). Let $u(x,t_m)=\langle \Lambda p, p\rangle$, $v(x,t_m)=\langle q,q\rangle$. Then:

(i) the flow equations (F_0^*), (F_m^*) are reduced to the spatial part and the time part, respectively, of the Lax pair for the higher-order CKdV equation:

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\Lambda+\frac{1}{2}u & -v \\ 1 & \frac{1}{2}\Lambda-\frac{1}{2}u \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \tag{3.2}$$

$$\begin{aligned}\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix} &= \sum_{j=0}^{m}\left[A(G_{j-1})\begin{pmatrix} \Lambda^{m-j}q \\ \Lambda^{m-j}p \end{pmatrix} - B(G_{j-1})\begin{pmatrix} \Lambda^{m+1-j}q \\ \Lambda^{m+1-j}p \end{pmatrix}\right] \\ &\quad + \sum_{j=1}^{m} c_j \sum_{k=1}^{m+1-j}\left[A(G_{k-2})\begin{pmatrix} \Lambda^{m+1-j-k}q \\ \Lambda^{m+1-j-k}p \end{pmatrix} - B(G_{k-2})\begin{pmatrix} \Lambda^{m+2-j-k}q \\ \Lambda^{m+2-j-k}p \end{pmatrix}\right].\end{aligned} \tag{3.3}$$

(ii) $u(x,t_m)$, $v(x,t_m)$ satisfy the higher-order CKdV equation

$$(u_{t_m}, v_{t_m})^{\mathrm{T}} = X_m + c_1X_{m-1}+\ldots+c_mX_0 \tag{3.4}$$

Proof. From the expression (2.8), we see that (3.2) holds, from which it is easy to calculate that

$$\begin{aligned}\langle\Lambda^{j-1}q,p\rangle &= \tfrac{1}{2}\langle\Lambda^{j-1}p,p\rangle' + \tfrac{1}{2}u\langle\Lambda^{j-1}p,p\rangle - \tfrac{1}{2}\langle\Lambda^{j}p,p\rangle \\ \langle\Lambda^{j}q,q\rangle &= \langle\Lambda^{j}q,p\rangle' + v\langle\Lambda^{j}p,p\rangle.\end{aligned} \tag{3.5}$$

In view of (2.6), (2.7), (3.5) and $\langle p,p\rangle=1$, we have

$$\begin{aligned}\begin{pmatrix} q_{tm} \\ p_{tm} \end{pmatrix} &= \begin{pmatrix} F^*_{m,p} \\ -F^*_{m,q} \end{pmatrix} = \begin{pmatrix} F_{m,p}-\mu_mF_p \\ -F_{m,q}+\mu_mF_q \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Lambda^{m+1}q \\ \Lambda^{m+1}p \end{pmatrix} + \begin{pmatrix} \frac{1}{2}u & -v \\ 1 & -\frac{1}{2}u \end{pmatrix}\begin{pmatrix} \Lambda^{m}q \\ \Lambda^{m}p \end{pmatrix} \\ &\quad + \sum_{j=2}^{m+1}\begin{pmatrix} \frac{1}{2}\langle\Lambda^{j-1}p,p\rangle'+\frac{1}{2}u\langle\Lambda^{j-1}p,p\rangle & -\langle\Lambda^{j-1}q,p\rangle'-v\langle\Lambda^{j-1}p,p\rangle \\ \langle\Lambda^{j-1}p,p\rangle & -\frac{1}{2}\langle\Lambda^{j-1}p,p\rangle'-\frac{1}{2}u\langle\Lambda^{j-1}p,p\rangle \end{pmatrix} \\ &\quad \times\begin{pmatrix} \Lambda^{m+1-j}q \\ \Lambda^{m+1-j}p \end{pmatrix} - \sum_{j=1}^{m}\begin{pmatrix} \frac{1}{2}\langle\Lambda^{j}p,p\rangle & 0 \\ 0 & -\frac{1}{2}\langle\Lambda^{j}p,p\rangle \end{pmatrix}\begin{pmatrix} \Lambda^{m+1-j}q \\ \Lambda^{m+1-j}p \end{pmatrix}.\end{aligned} \tag{3.6}$$

Substituting (2.10) into (3.6) yields (3.3). Through direct calculations and noticing $\langle q, p\rangle = 0$ and (2.10), we obtain

$$\begin{pmatrix} u_{t_m} \\ v_{t_m} \end{pmatrix} = 2\begin{pmatrix} \langle \Lambda p, p_{t_m}\rangle \\ \langle q, q_{t_m}\rangle \end{pmatrix} = 2\begin{pmatrix} \langle \Lambda p, F_{m,p}\rangle - \mu_m\langle \Lambda p, F_p\rangle \\ -\langle q, F_{m,q}\rangle + \mu_m\langle q, F_q\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \Lambda^{m+2}p, p\rangle + 2\langle \Lambda^{m+1}q, p\rangle - u\langle \Lambda^{m+1}p, p\rangle \\ v\langle \Lambda^{m+1}p, p\rangle - \langle \Lambda^{m+1}q, q\rangle \end{pmatrix} = J\begin{pmatrix} -\langle \Lambda^{m+1}q, p\rangle \\ \langle \Lambda^{m+1}p, p\rangle \end{pmatrix}$$

$$= J\left(G_m + \sum_{j=1}^{m+1} c_j G_{m-j} + \varepsilon_m G_{-2} \right) = X_m + c_1 X_{m-1} + \ldots + c_m X_0.$$

4. A completely integrable Bargmann system

Consider the Bargmann constraint

$$G_0 = \sum_{j=1}^{N} \nabla\lambda_j, \qquad \text{i.e.} \qquad u = \langle p, p\rangle, \quad v = -\langle q, p\rangle. \tag{4.1}$$

The nonlinearisation of (1.3) under (4.1) gives the Bargmann system

$$\text{(B)} \qquad \begin{cases} q' = -\frac{1}{2}\Lambda q + \frac{1}{2}\langle p, p\rangle q + \langle q, p\rangle p = \dfrac{\partial H}{\partial p} \\ p' = q + \frac{1}{2}\Lambda p - \frac{1}{2}\langle p, p\rangle p = -\dfrac{\partial H}{\partial q} \end{cases} \tag{4.2}$$

where the Hamiltonian function is

$$H = -\tfrac{1}{2}\langle \Lambda q, p\rangle + \tfrac{1}{2}\langle p, p\rangle\langle q, p\rangle - \tfrac{1}{2}\langle q, q\rangle.$$

Let

$$\Gamma_k = \sum_{\substack{j=1 \\ j\neq k}}^{N} \frac{B_{kj}^2}{\lambda_k - \lambda_j} \tag{4.3}$$

where $B_{kj} = p_k q_j - p_j q_k$, we have (see [1, 4, 5])

Lemma 4.1.

$$(\langle q, p\rangle, p_l^2) = 2p_l^2 \qquad (\langle q, p\rangle, q_l^2) = -2q_l^2 \tag{4.4}$$

$$(p_k^2, \Gamma_l) = \frac{-4B_{lk}}{\lambda_l - \lambda_k} p_k p_l \qquad (q_k p_k, \Gamma_l) = \frac{-2B_{lk}}{\lambda_l - \lambda_k}(p_k q_l + q_k p_l)$$

$$(q_k^2, \Gamma_l) = \frac{-4B_{lk}}{\lambda_l - \lambda_k} q_k q_l. \tag{4.5}$$

Lemma 4.2.

$$(\Gamma_k, \Gamma_l) = (\langle q, p\rangle, \Gamma_l) = (\langle q, p\rangle, q_l p_l) = 0 \tag{4.6}$$

$$(p_k^2, p_l^2) = (q_k^2, q_l^2) = (q_k p_k, q_l p_l) = 0 \tag{4.7}$$

$$(q_k p_k, p_l^2) = 2p_k p_l \delta_{kl} \qquad (q_k^2, p_l^2) = 4q_k p_l \delta_{kl} \qquad (q_k^2, p_l q_l) = 2q_k q_l \delta_{kl}. \tag{4.8}$$

All these results can be verified directly.

Theorem 4.3. $E_1, \ldots, E_N$ defined as follows compose an N-involutive system:

$$E_k = \tfrac{1}{2}\langle q, p\rangle p_k^2 - \tfrac{1}{2}q_k^2 - \tfrac{1}{2}\lambda_k q_k p_k - \tfrac{1}{2}\Gamma_k. \tag{4.9}$$

Proof. Obviously $(E_k, E_l)=0$ for $l=k$. Suppose $k\neq l$, in virtue of (4.6), (4.7) and the property of Poisson bracket in $(\mathbb{R}^{2N}, \mathrm{d}p\wedge \mathrm{d}q)$, we have

$$\begin{aligned}4(E_k, E_l) = &-\langle q,p\rangle(\Gamma_k, p_l^2)+\langle q,p\rangle(\Gamma_l, p_k^2)+(\Gamma_k, q_l^2)-(\Gamma_l, q_k^2)\\ &+\lambda_l(\Gamma_k, q_lp_l)-\lambda_k(\Gamma_l, q_kp_k)+\langle q,p\rangle p_l^2(p_k^2, \langle q,p\rangle)\\ &-\langle q,p\rangle p_k^2(p_l^2,\langle q,p\rangle)-p_k^2(\langle q,p\rangle, q_l^2)+p_l^2(\langle q,p\rangle, q_k^2).\end{aligned} \tag{4.10}$$

Substituting (4.4) and (4.5) into (4.10) yields $(E_k, E_l)=0$.

Consider a bilinear function $Q_z(\xi, \eta)$ on $\mathbb{R}^N$ and its partial-fraction expansion and laurent expansion:

$$Q_z(\xi,\eta)\triangleq\langle(z-\Lambda)^{-1}\xi,\eta\rangle=\sum_{k=1}^{N}\frac{\xi_k\eta_k}{z-\lambda_k}=\sum_{m=0}^{\infty}z^{-m-1}\langle\Lambda^m\xi,\eta\rangle.$$

The generating function of Γ_k is (see [4, 5])

$$\begin{vmatrix}Q_z(q,q) & Q_z(q,p)\\ Q_z(p,q) & Q_z(p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{\Gamma_k}{z-\lambda_k}.$$

Hence the generating function of E_k is

$$\begin{aligned}\mathscr{F}&=\tfrac{1}{2}\langle q,p\rangle Q_z(p,p)-\tfrac{1}{2}Q_z(q,q)-\tfrac{1}{2}Q_z(\Lambda q,p)-\tfrac{1}{2}\begin{vmatrix}Q_z(q,q) & Q_z(q,p)\\ Q_z(p,q) & Q_z(p,p)\end{vmatrix}\\ &=\sum_{k=1}^{N}\frac{E_k}{z-\lambda_k}.\end{aligned}$$

Theorem 4.4. The functions defined as follows are in involution in pairs, $(\bar{F}_k, \bar{F}_l)=0$,

$$\bar{F}_0=\tfrac{1}{2}\langle q,p\rangle\langle p,p\rangle-\tfrac{1}{2}\langle q,q\rangle-\tfrac{1}{2}\langle\Lambda q,p\rangle \tag{4.11}$$

$$\bar{F}_m=\tfrac{1}{2}\langle q,p\rangle\langle\Lambda^m p,p\rangle-\tfrac{1}{2}\langle\Lambda^m q,q\rangle-\tfrac{1}{2}\langle\Lambda^{m+1}q,p\rangle-\tfrac{1}{2}\sum_{j=1}^{m}\begin{vmatrix}\langle\Lambda^{j-1}q,q\rangle & \langle\Lambda^{j-1}q,p\rangle\\ \langle\Lambda^{m-j}p,q\rangle & \langle\Lambda^{m-j}p,p\rangle\end{vmatrix}. \tag{4.12}$$

Moreover,

$$\bar{F}_m=\sum_{k=1}^{N}\lambda_k^m E_k.$$

Proof. Substituting the Laurent expansion of Q_z into $\mathscr{F}$, we have $\mathscr{F}=\sum_{m=0}^{\infty}z^{-m-1}\bar{F}_m$. On the other hand, expanding $(z-\lambda_k)^{-1}$ as a power series in z^{-1}, we get

$$\mathscr{F}=\sum_{k=1}^{N}\frac{E_k}{\xi-\lambda_k}=\sum_{m=0}^{\infty}\xi^{-m-1}\sum_{k=1}^{N}\lambda_k^m E_k.$$

Thus $\bar{F}_m=\sum_{k=1}^{N}\lambda_k^m E_k$, and the involutivity of $\{E_k\}$ implies the involutivity of $\{\bar{F}_m\}$.

Theorem 4.5. The Hamiltonian system defined by the Bargmann system (4.2) $(\mathbb{R}^{2N}, \mathrm{d}p\wedge \mathrm{d}q, H=\bar{F}_0)$, is completely integrable in the Liouville sense.

Theorem 4.6. Let (q,p) be a solution of the Bargmann system (B), then $u=\langle p,p\rangle$, $v=-\langle q,p\rangle$ satisfy a stationary cKdV equation

$$X_N+\alpha_1X_{N-1}+\ldots+\alpha_NX_0=0 \tag{4.13}$$

with suitably chosen constants $\alpha_1,\ldots,\alpha_N$.

Proof. Operating with $(J^{-1}K)^k$ upon the first expression of (4.1), we have

$$G_k+\sum_{j=1}^{k} c_j G_{k-j-1}+\varepsilon_k G_{-2}=\sum_{j=1}^{N} \lambda_j^k \nabla \lambda_j \tag{4.14}$$

by virtue of (1.6) and (1.8). Noticing the polynomial (2.11), we obtain (4.13) in a similar manner to the proof of (2.9).

5. The involutive solutions of the cKdV equation associated with the Bargmann system

Consider the canonical system for the $\bar{F}_m$-flow $g_m^{t_m}$

$$(\bar{F}_m) \qquad \frac{\partial}{\partial t_m}\begin{pmatrix} q \\ p \end{pmatrix}=\begin{pmatrix} \partial \bar{F}_m/\partial p \\ -\partial \bar{F}_m/\partial q \end{pmatrix}=I\nabla \bar{F}_m. \tag{5.1}$$

Denote the flow variables of $(\bar{F}_0)$ and $(\bar{F}_m)$ by $x=t_0$ and $t=t_m$, respectively. Then the involutive solution of the consistent equations $(\bar{F}_0)$ and $(\bar{F}_m)$:

$$\begin{pmatrix} q(x, t_m) \\ p(x, t_m) \end{pmatrix}=g_0^x g_m^{t_m}\begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}$$

is a smooth function of (x, t_m) in view of the commutativity of the flows g_0^x and $g_m^{t_m}$

Theorem 5.1. Let $(q(x, t_m), p(x, t_m))^{\mathrm{T}}$ be an involutive solution of the consistent system $(\bar{F}_0)$, $(\bar{F}_m)$; let $u(x, t_m)=\langle p, p\rangle$ and $v(x, t_m)=-\langle q, p\rangle$. Then:

(i) the equations $(\bar{F}_0)$, $(\bar{F}_m)$ are reduced to the spatial part and the time part, respectively, of the Lax pair for the higher order cKdV equation

$$\begin{pmatrix} q' \\ p' \end{pmatrix}=\begin{pmatrix} -\frac{1}{2}\Lambda+\frac{1}{2}u & -v \\ 1 & \frac{1}{2}\Lambda-\frac{1}{2}u \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \tag{5.2}$$

$$\begin{aligned}\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix}=&\sum_{j=0}^{m}\left[A(G_{j-1})\begin{pmatrix} \Lambda^{m-j}q \\ \Lambda^{m-j}p \end{pmatrix}-B(G_{j-1})\begin{pmatrix} \Lambda^{m+1-j}q \\ \Lambda^{m+1-j}p \end{pmatrix}\right]\\ &+\sum_{j=1}^{m-1} c_j \sum_{k=1}^{m-j}\left[A(G_{k-2})\begin{pmatrix} \Lambda^{m-j-k}q \\ \Lambda^{m-j-k}p \end{pmatrix}-B(G_{k-2})\begin{pmatrix} \Lambda^{m+1-j-k}q \\ \Lambda^{m+1-j-k}p \end{pmatrix}\right]\end{aligned} \tag{5.3}$$

(ii) $u(x, t_m)=\langle p, p\rangle$ and $v(x, t_m)=-\langle q, p\rangle$ satisfy the higher order cKdV equation

$$(u_{t_m}, v_{t_m})^{\mathrm{T}}=X_m+c_1X_{m-2}+\ldots+c_{m-1}X_0. \tag{5.4}$$

Proof. Obviously (5.2) holds from (4.2) and (4.11). From the expressions (4.12) and (3.5), we have

$$\begin{aligned}\begin{pmatrix} q_{t_m} \\ p_{t_m} \end{pmatrix}=&\sum_{j=1}^{m}\begin{pmatrix} \frac{1}{2}\langle\Lambda^{j-1}p, p\rangle'+\frac{1}{2}u\langle\Lambda^{j-1}p, p\rangle & -\langle\Lambda^{j-1}q, p\rangle'-v\langle\Lambda^{j-1}p, p\rangle \\ \langle\Lambda^{j-1}p, p\rangle & -\frac{1}{2}\langle\Lambda^{j-1}p, p\rangle'-\frac{1}{2}u\langle\Lambda^{j-1}p, p\rangle \end{pmatrix}\begin{pmatrix} \Lambda^{m-j}q \\ \Lambda^{m-j}p \end{pmatrix}\\ &+\sum_{j=1}^{m}\begin{pmatrix} -\frac{1}{2}\langle\Lambda^{j-1}p, p\rangle & 0 \\ 0 & \frac{1}{2}\langle\Lambda^{j-1}p, p\rangle \end{pmatrix}\begin{pmatrix} \Lambda^{m+1-j}q \\ \Lambda^{m+1-j}p \end{pmatrix}+\begin{pmatrix} \frac{1}{2}u & -v \\ 1 & -\frac{1}{2}u \end{pmatrix}\begin{pmatrix} \Lambda^{m}q \\ \Lambda^{m}p \end{pmatrix}\\ &+\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Lambda^{m+1}q \\ \Lambda^{m+1}p \end{pmatrix}.\end{aligned}$$

Substituting (4.14) into (5.5) yields (5.3). Through direct calculations we have

$$\begin{pmatrix} u_{t_m} \\ v_{t_m} \end{pmatrix} = \begin{pmatrix} 2\langle p, p_{t_m}\rangle \\ -\langle q_{t_m}, p\rangle - \langle q, p_{t_m}\rangle \end{pmatrix} = \begin{pmatrix} \langle \Lambda^{m+1}p, p\rangle - u\langle \Lambda^m p, p\rangle + 2\langle \Lambda^m q, p\rangle \\ v\langle \Lambda^m p, p\rangle - \langle \Lambda^m q, q\rangle \end{pmatrix}$$

$$= J\begin{pmatrix} -\langle \Lambda^m q, p\rangle \\ \langle \Lambda^m p, p\rangle \end{pmatrix} = J\left(G_m + \sum_{j=1}^{m} c_j G_{m-j-1} + \varepsilon_m G_{-2} \right)$$

$$= X_m + c_1 X_{m-2} + \ldots + c_{m-1} X_0$$

in view of (4.14).

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